Functions

Limits (part 2)

Introduction

In the previous video, we’ve learned about the notion of limit and how to find it graphically.

In this video we will learn how to find it by calculation.

Operations on the limits

Given: $\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to a} g(x) = l'$ (a, l and l’ can be ∞ or a real number)

$$\lim_{x\to a}\left(f(x)+g(x)\right) =$$

If $l\,;l' \epsilon\ IR$	$l + l'$
If $l\ \epsilon\ IR$ and $l' = \pm\infty$	$l \pm \infty = \pm\infty$
If $l = l' = +\infty$	$+\infty + \infty = +\infty$
If $l = l' = -\infty$	$-\infty - \infty = -\infty$
If $l = +\infty\,;l' = -\infty$	$+\infty - \infty$ indeterminate form

Operations on the limits

$\lim_{x\to a}\left(f(x)\times g(x)\right) =$

If $l\,;l' \epsilon\, IR^*$	$l\times l'$
If $l\ \epsilon\ IR$ and $l' = \pm\infty$	$l\times(\pm\infty) = \pm\infty$
If $l = l' = +\infty$	$(+\infty)\times(+\infty) = +\infty$
If $l = l' = -\infty$	$(-\infty)\times(-\infty) = +\infty$
If $l = +\infty\,; l' = -\infty$	$(+\infty)\times(-\infty) = -\infty$
If $l = 0; l' = \pm\infty$	$0\times(\pm\infty)$ indeterminate form

Operations on the limits

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) =$$

If $l\,; l' \epsilon\, IR^*$	$\frac{l}{l\prime}$
If $l \epsilon\, IR$ and $l' = \pm\infty$	$\frac{l}{\pm\infty} = 0$
If $l = \pm\infty$ and $l' \epsilon\, IR$	$\frac{\pm\infty}{l'} = \pm\infty$
If $l = l' = \pm\infty$	$\frac{\pm\infty}{\pm\infty}$ indeterminate form
If $l = 0; l' \epsilon\, IR^*$	$\frac{0}{l\prime} = 0$
If $l = l; l' = 0$	$\frac{l}{0} = \pm\infty$
If $l = l' = 0$	$\frac{0}{0}$ indeterminate form

Operations on the limits

$$\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x) = kl$$

$$\lim_{x\to a}(f(x) \pm k) = \lim_{x\to a} f(x) \pm k = l \pm k$$

$$\lim_{x\to\pm\infty} \frac{k}{f(x)} = \frac{k}{l} \quad ; \quad l \neq 0$$

$$\lim_{x\to\pm\infty} \sqrt{f(x)} = \sqrt{l} \ ; \ l \geq 0$$

Why indeterminate forms???

$(+\infty - \infty)$ or $(-\infty + \infty)$

We don’t know the exact value of $+\infty$ or $-\infty$

$0 \times (\pm\infty)$

We don’t know which rule it follows:

$0 \times$anything is 0

Or

Number × ∞ is ∞

Why indeterminate forms???

$$\frac{0}{0}$$

We don’t know which rule it follows:

$\frac{0}{number}$ is 0

Or

$\frac{number}{0}$ is ∞

$$\frac{\pm\infty}{\pm\infty}$$

We don’t know the exact value of +∞ or -∞ so the answer can be 1, -1 or less than 0 or more than 0…

How to find limit at infinity by calculation?

Information to be used:

$$\lim_{x \to +\infty} x = +\infty$$

$$\lim_{x \to -\infty} x = -\infty$$

$$\lim_{x \to +\infty} x^n = +\infty$$

$$\lim_{x \to -\infty} x^n = \begin{cases} +\infty & \text{if n is even} \\ -\infty & \text{if n is odd} \end{cases}$$

$\lim\limits_{x \to \pm\infty} kx^n = \pm\infty$ according to the sign of the real number k and whether n is even or odd

How to find limit at infinity by calculation?

- If f is a polynomial: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_0$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} a_n x^n$$

Leading term of f

Example:

❶ $f(x) = 2x + 3$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} 2x = 2(+\infty) = +\infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2x = 2(-\infty) = -\infty$$

How to find limit at infinity by calculation?

- If f is a polynomial: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_0$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n$$

Example:

❷ $f(x) = -2x + 3$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} -2x = -2(+\infty) = -\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} -2x = -2(-\infty) = +\infty$$

How to find limit at infinity by calculation?

- If f is a polynomial: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_0$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n$$

Example:

❸ $f(x) = 3x^2 + 4x - 5$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} 3x^2 = 3(+\infty)^2 = 3(+\infty) = +\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} 3x^2 = 3(-\infty)^2 = 3(+\infty) = +\infty$$

How to find limit at infinity by calculation?

- If f is a polynomial: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_0$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n$$

Example:

❹ $f(x) = -3x^2 + 4x - 5$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} -3x^2 = -3(+\infty)^2 = -3(+\infty) = -\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} -3x^2 = -3(-\infty)^2 = -3(+\infty) = -\infty$$

How to find limit at infinity by calculation?

- If f is a polynomial: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_0$

$$lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} a_n x^n$$

Example:

❺ $f(x) = -x^3 - 5x^2 + 1$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} -x^3 = -(+\infty)^3 = -(+\infty) = -\infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -x^3 = -(-\infty)^3 = -(-\infty) = +\infty$$

How to find limit at infinity by calculation?

❖ If f is a rational: $f(x) = \frac{P(x)}{Q(x)}$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{\text{leading term of } P(x)}{\text{leading term of } Q(x)}$$

Example:

❶ $f(x) = \frac{x^2+1}{2x+3}$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2}{2x} = \lim_{x\to+\infty} \frac{x}{2} = \frac{+\infty}{2} = +\infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{2} = \frac{-\infty}{2} = -\infty$$

How to find limit at infinity by calculation?

- ❖ If f is a rational: $f(x) = \frac{P(x)}{Q(x)}$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{\text{leading term of } P(x)}{\text{leading term of } Q(x)}$$

Example:

❷ $f(x) = \frac{x^3}{-3x^3+2}$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^3}{-3x^3} = -\frac{1}{3}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3}{-3x^3} = -\frac{1}{3}$$

How to find limit at infinity by calculation?

- If f is a rational: $f(x) = \frac{P(x)}{Q(x)}$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{\text{leading term of } P(x)}{\text{leading term of } Q(x)}$$

Example:

③ $f(x) = \frac{-5x^3+2x-4}{x^6+1}$ $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{-5x^3}{x^6} = \lim_{x \to +\infty} -\frac{5}{x^3} = -\frac{5}{+\infty} = 0$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{-5x^3}{x^6} = \lim_{x \to -\infty} -\frac{5}{x^3} = -\frac{5}{-\infty} = 0$$

How to find limit at infinity by calculation?

- As a result, if f is rational

$$\mathit{lim}_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{\text{leading term of } P(x)}{\text{leading term of } Q(x)}$$

$$= \begin{cases} k & if \deg(P) = \deg(Q) \\ 0 & if \deg(P) < \deg(Q) \\ \pm\infty & if \deg(P) > \deg(Q) \end{cases}$$

How to find limit at infinity by calculation?

❖ If f is a irrational (including radical):

Example:

❶ $f(x) = \sqrt{x-1}$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \sqrt{x} = \sqrt{+\infty} = +\infty$$

$$\lim_{x\to-\infty} f(x) = \nexists$$

How to find limit at infinity by calculation?

- If f is a irrational (including radical):

Example:

❷ $f(x) = \sqrt{x^2 - 1}$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \sqrt{x^2} = \lim_{x \to +\infty} |x| = |+\infty| = +\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} |x| = |-\infty| = +\infty$$

How to find limit at infinity by calculation?

- ❖ If f is a irrational (including radical):

Example:

❸ $f(x) = \sqrt{x} + x$

$$\lim_{x \to +\infty} f(x) = \sqrt{+\infty} + \infty = +\infty + \infty = +\infty$$

$$\lim_{x \to -\infty} f(x) = \nexists$$

How to find limit at infinity by calculation?

❖ If f is a irrational (including radical):

Example:

❹ $f(x) = \sqrt{x} - x$

$$\lim_{x \to +\infty} f(x) = \sqrt{+\infty} - \infty = +\infty - \infty$$

Rationalize

$$\lim_{x \to +\infty} \sqrt{x} - x \times \frac{\sqrt{x}+x}{\sqrt{x}+x} = \lim_{x \to +\infty} \frac{x-x^2}{\sqrt{x}+x} = \lim_{x \to +\infty} \frac{-x^2}{\sqrt{x}+x} = \lim_{x \to +\infty} \frac{-x^2}{x\left(\frac{\sqrt{x}}{x}+1\right)}$$

$$= \lim_{x \to +\infty} \frac{-x}{\frac{1}{\sqrt{x}}+1} = \frac{-\infty}{\frac{1}{+\infty}+1} = \frac{-\infty}{0+1} = -\infty$$

$$\lim_{x \to -\infty} f(x) = \nexists$$

How to find limit at infinity by calculation?

❖ If f is a irrational (including radical):

Example:

❺ $f(x) = \sqrt{\overset{+\infty}{x^2} + 1} + \overset{+\infty}{x}$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \sqrt{x^2} + x = \lim_{x\to+\infty} |x| + x = \lim_{x\to+\infty} x + x = \lim_{x\to+\infty} 2x$$

$$= 2(+\infty) = +\infty$$

How to find limit at infinity by calculation?

❖ If f is a irrational (including radical):

Example:

❺ $f(x) = \sqrt{\overset{+\infty}{x^2} + 1} + \overset{-\infty}{x}$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \sqrt{x^2+1} + x \quad \times \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}-x} = \lim_{x\to-\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}+x} = \lim_{x\to-\infty} \frac{1}{\sqrt{x^2}+x} = \lim_{x\to-\infty} \frac{1}{x+x}$$

$$= \lim_{x\to-\infty} \frac{1}{2x} = \frac{1}{-\infty} = 0$$

Time for practice

Calculate the limits of f at ±∞.

❶ $f(x) = -3x^2 + 2x + 4$ $\lim\limits_{x\to\pm\infty} f(x) = -\infty$

❷ $f(x) = \frac{1}{2}x^7 - 3x^5 + 2x^3 + 5$ $\lim\limits_{x\to+\infty} f(x) = +\infty$; $\lim\limits_{x\to-\infty} f(x) = -\infty$

❸ $f(x) = (x+1)(x^5 + 3x)$ $\lim\limits_{x\to\pm\infty} f(x) = +\infty$

❹ $f(x) = \frac{1}{\sqrt{x}+5}$ $\lim\limits_{x\to+\infty} f(x) = 0$; $\lim\limits_{x\to-\infty} f(x) = \nexists$

❺ $f(x) = \frac{x^5+2x+2}{x^3+2x}$ $\lim\limits_{x\to\pm\infty} f(x) = +\infty$

❻ $f(x) = \frac{2x(x+3)}{(x+1)^2}$ $\lim\limits_{x\to\pm\infty} f(x) = 2$